

THE CHAOS GAME ON AN ITERATED FUNCTION SYSTEM FROM A TOPOLOGICAL POINT OF VIEW

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ABSTRACT. We investigate combinatorial issues relating to the use of random orbit approximations to the attractor of an iterated function system with the aim of clarifying the role of the stochastic process during generation the orbit. A Baire category counterpart of almost sure convergence is presented; and a link between topological and probabilistic methods is observed.

1. INTRODUCTION

We prove that the chaos game, for all but a σ -porous set of orbits, yields a set that intersects all of the fibres of an attractor A of a general iterated function system (IFS). The IFS may not be contractive and may possess multiple attractors. In [6] it was shown that, in proper metric spaces, attractors are limits of certain non-stationary stochastic chaos games; this generalized the canonical explanation, based on stationary stochastic processes, [13], of why the chaos game works to generate attractors. Here we present different results, based primarily in topology and category rather than in stochastic processes. Also, our results may have implications on how data strings are analyzed, as we explain next.

An iterated function system $F = (X, f_\sigma : \sigma \in \Sigma)$ is a finite set of discrete dynamical systems $f_\sigma : X \rightarrow X$. If $(\sigma_k)_{k=1}^\infty$ is a sequence in Σ then the corresponding chaos game orbit [1, p.2 and p.91] of a point $x_0 \in X$ is the sequence $(x_k)_{k=0}^\infty$ defined iteratively by $x_k = f_{\sigma_k}(x_{k-1})$ for $k = 1, 2, \dots$. The chaos game may be used (i) in computer graphics, to render pictures of fractals and other sets [2, 4, 31], and (ii) in data analysis to reveal patterns in long data strings such as DNA base pair sequences, see for example papers that cite [20]. If the maps f_σ are contractions on a complete metric space X , and if the sequence $(\sigma_k)_{k=1}^\infty$ is suitably chaotic or random, then the tail of $(x_k)_{k=0}^\infty$ converges to the unique attractor of F . In applications to computer graphics, long finite strings $(\sigma_k)_{k=1}^L$ are used, say with $L = 10^9$. In applications to genome analysis, if $(\sigma_k)_{k=1}^L$ is a long finite sequence, say $L = 2.9 \times 10^9$ for the number of base pairs in human DNA, and if the attractor of F is a simple geometrical object such as a square, then $(x_k)_{k=0}^L$ may be plotted, yielding a "picture" of $(\sigma_k)_{k=1}^L$. Such pictures may be used to identify patterns in $(\sigma_k)_{k=1}^L$, and used, for example, to distinguish different types of DNA, [20]. In the first case (i) a stochastic process is used to define the chaos game orbit $(x_k)_{k=0}^L$ and to describe the attractor A of the IFS. In the second case (ii) a deterministic process, specified by a given data string, is used to define the chaos game orbit $(x_k)_{k=0}^L$; how this orbit sits in the attractor, that is, *the relationship between the*

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deterministic orbit and the stochastic orbit, provides the pattern or signature of the string. Notice that there are two types of chaos game here: one describing an attractor, and the other describing a data string. Our results suggest the feasibility of data analysis (a) using topological concepts (b) using strongly-fibred IFSs.

The type of IFS F that we consider is quite general: the only restrictions are that the underlying metric space X is complete and the functions $f_\sigma : X \rightarrow X$ are continuous. In Section 2 we define an attractor, its basin of attraction, and chaos games. In Section 2 we also define the fibres of an attractor and describe how attractors are classified according to their fibre structure. The types of fibre structure of an attractor are minimal-fibred, strongly-fibred and point-fibred. In contrast to the situation for a contractive IFS, as in the classical Hutchinson theory, see [18], it is not generally possible to associate a continuous map from the code space Σ^∞ onto an attractor. Consequently, results concerning the behaviour of the chaos game cannot be inferred from analogous results, on the code space itself, by continuous projection onto the attractor. Nonetheless, in Section 3, we establish Theorem 1, which says that the tail of any disjunctive chaos game orbit, starting from any point in the basin of an attractor, converges in the Hausdorff metric to a set C_∞ that is both contained in the attractor and contains a point belonging to each fibre of the attractor. This is achieved via a sequence of lemmas, similar to ones in [6], but replacing stochastic sequences by disjunctive ones, and lifting the requirement that X be proper. Theorem 1 allows us to prove in Section 4 that the chaos game, starting from any point in the basin of strongly-fibred attractor, yields the attractor, except for a set of strings that is small in the sense of Baire category; specifically Theorem 4 says that the set of strings for which the chaos game does not converge to the strongly-fibred attractor is σ -porous, which is stronger than first category. In Section 5, we define the notion of a disjunctive stochastic process, which generalizes the notion of a chain with complete connections [32]; then we prove, *as a consequence the foregoing material*, that Theorem 5 holds: namely, a chaos game produced by disjunctive stochastic process converges to a strongly-fibred attractor almost surely. Thus, we see that the stochastic version is a limiting consequence of combinatorics and topology, as it should be.

Finally, in Section 6 we establish Theorem 6 – the Rapunzel Theorem – which illustrates the power of the disjunctiveness in the chaos game algorithm in the commonly occurring situation where an IFS of homeomorphisms on a compact metric space possesses a unique point-fibred attractor A and a unique point-fibred repeller A^* . This situation occurs for Möbius IFSs on the Riemann sphere [36]. Basically, the result says that if $(\sigma_k)_{k=1}^\infty$ is a disjunctive sequence, then even when the point x_0 belongs to the dual repeller A^* , the “usual/typical/almost always” event is that the chaos game orbit “escapes from the tower”, the disjunctive sequence “lets down her hair” and the sequence of points in the chaos game orbit dances out of the clutches of the dual repeller. Why is this surprising? For a number of reasons, but mainly this: A^* is the complement of the basin of attractor A , so it is not true that $\lim_{k \rightarrow \infty} F^k(\{x\}) = A$ for $x \in A^*$, and A^* may have nonempty interior.

2. DEFINITIONS

Throughout, let (X, d) be a complete metric space with metric d . For $b \in X$, $C \subset X$ we denote

$$d(b, C) := \inf_{c \in C} d(b, c),$$

and for $B \subset X$, $\varepsilon > 0$

$$N_\varepsilon B := \{x \in X : d(x, B) < \varepsilon\}.$$

The **Hausdorff distance** between $B, C \subset X$ is defined as

$$h(B, C) := \inf\{r > 0 : B \subset N_r C, C \subset N_r B\}.$$

Let $\mathcal{K}(X)$ denote the set of nonempty compact subsets of X . Then $(\mathcal{K}(X), h)$ is also a complete metric space, and may be referred to as a **hyperspace** ([1, 2, 8, 14, 17]).

The system $F = (X, f_\sigma : \sigma \in \Sigma)$, comprising a finite set of continuous maps $f_\sigma : X \rightarrow X$, is called an **iterated function system** (IFS) on X [3]. Without risk of ambiguity we use the same notation F for the IFS and the associated **Hutchinson operator**

$$F : \mathcal{K}(X) \rightarrow \mathcal{K}(X) \ni B \longmapsto F(B) = \bigcup_{\sigma \in \Sigma} f_\sigma(B) = \{f_\sigma(b) : \sigma \in \Sigma, b \in B\}.$$

This map is well-defined because the f_σ are continuous and finite unions of continuous images of compacta are compacta. Furthermore, it is a basic fact that $F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is continuous; a proof can be found in [5]. The k -fold composition of F is written as F^k .

Following [7] we say that $A \in \mathcal{K}(X)$ is an **attractor** of the IFS F on X when there exists an open neighbourhood $U(A) \supset A$ such that, in the metric space $(\mathcal{K}(X), h)$,

$$(2.1) \quad F^k(B) \underset{k \rightarrow \infty}{\longrightarrow} A, \text{ for } U(A) \supset B \in \mathcal{K}(X).$$

The union $\mathcal{B}(A)$ of all open neighborhoods $U(A)$ such that (2.1) is true is called the **basin** of A . Since $F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is continuous it follows that A is an invariant set for F , i.e. $A = F(A)$. Clearly, A is the unique fixed point of F in the basin of A , i.e., if $B = \bigcup_{\sigma \in \Sigma} f_\sigma(B)$ and $\mathcal{B}(A) \supset B \in \mathcal{K}(X)$, then $B = A$.

The **coordinate map** $\pi : \Sigma^\infty \rightarrow \mathcal{K}(A)$ for A (w.r.t. F) is defined by

$$\pi(\rho) = \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \dots \circ f_{\rho_K}(A) =: A_\rho$$

for all $\rho \in \Sigma^\infty$. The set A_ρ is called a **fibre** of A . If A_ρ is a singleton for all $\rho \in \Sigma^\infty$, then A is said to be **point-fibred**. A is **strongly-fibred** means that if \mathcal{U} is an open cover of A and $U \in \mathcal{U}$ then there is $\rho \in \Sigma^\infty$ such that $A_\rho \subset U$. For reasons related to a more general notion of "attractor", all attractors of IFSs are said to be **minimally-fibred**. Strongly-fibred is weaker than point-fibred which is weaker than the situation where A is the attractor of a contractive IFSs. Classification of attractors according their fibration is discussed in [28, Chapter 4].

Let $(\sigma_1, \sigma_2, \dots) \in \Sigma^\infty$. The associated orbit of $x_0 \in U(A)$ under F is the sequence $(x_k)_{k=0}^\infty$ defined by

$$(2.2) \quad \begin{cases} x_0 \in U(A), \\ x_k := f_{\sigma_k}(x_{k-1}), \ k \geq 1. \end{cases}$$

If $(\sigma_1, \sigma_2, \dots)$ is chosen according to some stochastic process, then $(x_k)_{k=0}^\infty$ is referred to as a **random orbit**. More generally, such orbits are referred to as **chaos game** orbits, see [1] and [33], for example.

We use the notation $f_w := f_{\sigma_1} \circ \dots \circ f_{\sigma_k}$ for a finite word $w = (\sigma_1, \dots, \sigma_k) \in \Sigma^k$, so that $x_k = f_w(x_0)$. The concatenation of two words $u = (v_1, \dots, v_m) \in \Sigma^m$

and $w = (\sigma_1, \dots, \sigma_k) \in \Sigma^k$ is $uw := (v_1, \dots, v_m, \sigma_1, \dots, \sigma_k) \in \Sigma^{m+k}$. Notice that $f_{uw} = f_u \circ f_w$. We may omit the parentheses and commas; for example $u = v_1 \dots v_m$.

3. MAIN IDEA

Throughout this section let $F = (X, f_\sigma : \sigma \in \Sigma)$ be an IFS with attractor $A \in \mathcal{K}(X)$ and basin $\mathcal{B}(A)$. Let $(x_k)_{k=0}^\infty$ denote the orbit of x_0 under F , associated with $(\sigma_1, \sigma_2, \dots) \in \Sigma^\infty$.

The following observation lies at the heart of this investigation. It is hidden in [6]; compare also with [22, Theorem 12.8.2].

Lemma 1. *Given $x_0 \in \mathcal{B}(A)$, we have $y \in A$ if and only if, for given $\varepsilon > 0$ there exists a natural number m and a word $w = (\sigma_m, \sigma_{m-1}, \dots, \sigma_1) \in \Sigma^m$ such that $d(f_w(x_0), y) < \varepsilon$.*

Proof. Suppose $y \in A$ and let $\varepsilon > 0$ be given. The definition of attractor implies that there exists an iteration m such that $h(F^m(\{x_0\}), A) < \varepsilon$, and in particular

$$y \in A \subset N_\varepsilon F^m(\{x_0\}).$$

But

$$N_\varepsilon F^m(\{x_0\}) = N_\varepsilon \bigcup_{w \in \Sigma^m} f_w(\{x_0\}) = \bigcup_{w \in \Sigma^m} N_\varepsilon f_w(\{x_0\}).$$

It follows that $y \in N_\varepsilon f_w(\{x_0\})$ for some $w \in \Sigma^m$. It follows that there exists a word $w = (\sigma_m, \sigma_{m-1}, \dots, \sigma_1) \in \Sigma^m$ such that $d(f_w(x_0), y) < \varepsilon$.

Conversely, suppose y is such that, given $\varepsilon > 0$, there exists a natural number m and a sequence $w = (\sigma_m, \sigma_{m-1}, \dots, \sigma_1) \in \Sigma^m$ with $d(f_w(x_0), y) < \varepsilon$. It follows that $d(y, F^m(\{x_0\})) < \varepsilon$. It follows that $y \in \lim_{m \rightarrow \infty} F^m(\{x_0\}) = A$. \square

For $\sigma \in \Sigma^\infty$, $x_0 \in X$, $k \in \{1, 2, \dots\}$, define

$$x_k := x_k(x_0, \sigma) := f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0).$$

For all $K = 0, 1, 2, \dots$ define

$$C_K := C_K(x_0, \sigma) := \overline{\bigcup_{k=K}^\infty \{x_k\}}.$$

It is straightforward to prove that $\{x_k\}_{k=0}^\infty$ is totally bounded; consequently $\{C_K\}_{K=0}^\infty$ is a decreasing (nested) sequence of nonempty compact sets that converges in the Hausdorff metric to a unique nonempty compact limit

$$C_\infty := C_\infty(x_0, \sigma) := \bigcap_{K=1}^\infty C_K.$$

Lemma 2. *If A is an attractor of F , $\mathcal{B}(A)$ is the basin of A , $x_0 \in \mathcal{B}(A)$ and $\sigma \in \Sigma^\infty$, then*

$$C_\infty(x_0, \sigma) \subset A.$$

Proof. First, it follows from Lemma 1 that $a \in A$ if, and only if, there is an infinite subsequence $\{k_l\}_{l=1}^\infty$ of $\{k\}_{k=1}^\infty$ and $\rho^{(k_l)} \in \Sigma^{k_l}$ for $l = 0, 1, 2, \dots$, such that

$$\{f_{\rho_{k_l}^{(k_l)}} \circ \dots \circ f_{\rho_{k_1}^{(k_1)}}(x_0)\}_{l=1}^\infty$$

converges to a , namely

$$\lim_{l \rightarrow \infty} f_{\rho_{k_l}^{(k_l)}} \circ \dots \circ f_{\rho_{k_1}^{(k_1)}}(\{x_0\}) = a.$$

Second, note that if $c \in C_\infty$, then there is an infinite subsequence $\{k_m\}_{m=1}^\infty$ of $\{k\}_{k=1}^\infty$ such that $\{f_{\sigma_{k_m}} \circ \dots \circ f_{\sigma_1}(x_0)\}_{m=0}^\infty$ converges to c . By the first observation, on choosing $\rho^{(k_l)} = \sigma_{k_l} \dots \sigma_1$ for $l = 1, 2, \dots$, we obtain $c \in A$. \square

The following lemma is perhaps surprising.

Lemma 3. *Let A be an attractor of F , let $\mathcal{B}(A)$ be the basin of A , let $\sigma \in \Sigma^\infty$ and let $x_0 \in \mathcal{B}(A)$. We have*

$$F(C_\infty(x_0, \sigma)) := \bigcup_{f \in F} f(C_\infty(x_0, \sigma)) \supset C_\infty(x_0, \sigma)$$

Proof. We have

$$\begin{aligned} F(C_K(x_0, \sigma)) &= \bigcup_{f \in F} f(C_K(x_0, \sigma)) \\ &= \bigcup_{f \in F} \overline{\bigcup_{k=K}^\infty \{f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \\ &= \bigcup_{f \in F} \overline{\bigcup_{k=K}^\infty \{f \circ f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \\ &\supset C_{K+1}(x_0, \sigma) \end{aligned}$$

We know that $F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is continuous. Taking limits of decreasing sequences, we obtain

$$F(C_\infty(x_0, \sigma)) \supset C_\infty(x_0, \sigma).$$

\square

In summary, so far, we have that for all $x_0 \in \mathcal{B}(A)$, for all $\sigma \in \Sigma^\infty$,

$$C_\infty(x_0, \sigma) \subset F(C_\infty(x_0, \sigma)) \subset A.$$

Lemma 4. *Let A be an attractor of F , let $\mathcal{B}(A)$ be the basin of A , let $x_0 \in \mathcal{B}(A)$, $\sigma \in \Sigma^\infty$, and let $\theta_1 \theta_2 \dots \theta_P \in \Sigma^P$ for some $P \in \{1, 2, \dots\}$. If*

$$\sigma_{M+1} \dots \sigma_{M+P} = \theta_1 \theta_2 \dots \theta_P$$

for infinitely many distinct positive integers M , then

$$f_{\theta_P} \circ \dots \circ f_{\theta_1}(C_\infty(x_0, \sigma)) \cap C_\infty(x_0, \sigma) \neq \emptyset.$$

Proof. Let $P = 1$. We have, for all positive integers K and L ,

(3.1)

$$\begin{aligned} f_{\theta_1}(C_K(x_0, \sigma)) \cap C_{K+L}(x_0, \sigma) &= f_{\theta_1} \left(\overline{\bigcup_{k=K}^\infty \{f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \right) \cap \left(\overline{\bigcup_{k=K+L}^\infty \{f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \right) \\ &= \left(\overline{\bigcup_{k=K}^\infty \{f_{\theta_1} \circ f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \right) \cap \left(\overline{\bigcup_{k=K+L}^\infty \{f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}} \right) \\ &\supset \overline{\bigcup_{\substack{k \in \{K+L, \dots\} \\ s.t. \sigma_k = \theta_1}} \{f_{\sigma_k} \circ \dots \circ f_{\sigma_1}(x_0)\}}. \end{aligned}$$

The last expression is nonempty because $\sigma_k = \theta_1$ for infinitely many values of k . It follows that $\{f_{\theta_1}(C_K) \cap C_{K+L}\}_{L=1}^{\infty}$ is a decreasing sequence of nonempty compact sets. It converges to a nonempty compact set and it converges to $f_{\theta_1}(C_K) \cap C_{\infty}$ so

$$f_{\theta_1}(C_K) \cap C_{\infty} \neq \emptyset$$

for all $K = 1, 2, \dots$. But now $\{f_{\theta_1}(C_K) \cap C_{\infty}\}_{K=1}^{\infty}$ is a decreasing sequence of nonempty sets and it converges to

$$f_{\theta_1}(C_{\infty}) \cap C_{\infty} \neq \emptyset.$$

This proves the result for the case $P = 1$. For the general case, replace f_{θ_1} by $f_{\theta_P} \circ \dots \circ f_{\theta_1}$ and adjust the expressions in (3.1) accordingly. \square

We say that the infinite word $\sigma = (\sigma_1, \sigma_2, \dots) \in \Sigma^{\infty}$ is **disjunctive** ([11, 35]) if it contains all possible finite words i.e.

$$\forall_m \forall_{w \in \Sigma^m} \exists_j \forall_{l=1, \dots, m} \sigma_{(j-1)+l} = w_l.$$

In fact any finite word appears in a disjunctive sequence of symbols infinitely often, because it reappears as part of longer and longer words.

Proposition 1. *The sequence $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ is disjunctive if and only if*

$$(3.2) \quad \forall_{n,m} \forall_{(\tau_1, \tau_2, \dots, \tau_m) \in \Sigma^m} \exists_{k \geq n} \forall_{l=1, \dots, m} \tau_l = \sigma_{k+l}.$$

Example 1. (*Champernowne sequence*). Let us write down finite words over the alphabet Σ : first the one-letter words, second two-letter words etc. An infinite word made by concatenating this list creates a disjunctive sequence of symbols in Σ^{∞} , a **Champernowne sequence**. Note that all normal sequences are disjunctive but the converse is not true.

Applications of disjunctive sequences in complexity, automata theory and number theory are described in the papers cited in [11].

What does disjunctiveness give us? Let \mathcal{S}_F denote the semigroup of continuous functions from X to itself, generated by F . That is

$$\mathcal{S}_F := \{f_{\sigma_1} \circ \dots \circ f_{\sigma_k} : k \in \{1, 2, \dots\}, \sigma_1 \dots \sigma_k \in \Sigma^k\}$$

where the semigroup operation is function composition.

Lemma 5. *Let A be an attractor of F , let $\mathcal{B}(A)$ be the basin of A , let $x_0 \in B$, and let $\sigma \in \Sigma^{\infty}$ be disjunctive. If $f \in \mathcal{S}_F$, then*

$$f(C_{\infty}(x_0, \sigma)) \cap C_{\infty}(x_0, \sigma) \neq \emptyset.$$

Proof. This is an immediate consequence of Lemma 4 combined with disjunctiveness of σ . \square

Theorem 1. *Let A be an attractor of F , let $\mathcal{B}(A)$ be the basin of A , let $x_0 \in \mathcal{B}(A)$ and let $\sigma \in \Sigma^{\infty}$ be disjunctive. The set $C_{\infty}(x_0, \sigma)$ intersects every fibre of A ; that is,*

$$A_{\rho} \cap C_{\infty}(x_0, \sigma) \neq \emptyset$$

for all $\rho \in \Sigma^{\infty}$.

Proof. We have

$$\begin{aligned} C_\infty \cap A_\rho &= C_\infty \cap \lim_{K \rightarrow \infty} f_{\rho_1} \circ \dots \circ f_{\rho_K}(A) \\ &= C_\infty \cap \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \dots \circ f_{\rho_K}(A) \text{ because decreasing,} \\ &= \bigcap_{K=1}^{\infty} (C_\infty \cap f_{\rho_1} \circ \dots \circ f_{\rho_K}(A)) \text{ easily checked,} \end{aligned}$$

But, since $C_\infty \subset A$ by Lemma 2, we have

$$C_\infty \cap f_{\rho_1} \circ \dots \circ f_{\rho_K}(A) \supset C_\infty \cap f_{\rho_1} \circ \dots \circ f_{\rho_K}(C_\infty)$$

for all K . Also, by Lemma 3 and the assumption that σ is disjunctive, we have

$$C_\infty \cap f_{\rho_1} \circ \dots \circ f_{\rho_K}(C_\infty) \neq \emptyset$$

for all K . It follows that $\{C_\infty \cap f_{\rho_1} \circ \dots \circ f_{\rho_K}(A)\}$ is a decreasing (nested) sequence of non-empty compact sets. It follows that

$$C_\infty \cap A_\rho = C_\infty \cap \bigcap_{K=1}^{\infty} f_{\rho_1} \circ \dots \circ f_{\rho_K}(A) \neq \emptyset.$$

□

This says that, given any fibre A_ρ of an attractor, there exists $p \in A_\rho$ and a subsequence of $\{x_k\}$ that converges to p .

Corollary 1. *Let A be an attractor of F , let B be the basin of A , let $x_0 \in \mathcal{B}(A)$ and let $\sigma \in \Sigma^\infty$ be disjunctive. If A is strongly-fibred, then*

$$C_\infty(x_0, \sigma) = A.$$

That is, the tails of the random orbit

$$(3.3) \quad \{x_n : n \geq p\} \xrightarrow[p \rightarrow \infty]{} A$$

converge to the attractor with respect to the Hausdorff distance, and

$$(3.4) \quad A = \overline{\bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} \{x_n\}}.$$

Proof. Let \mathcal{U} be a cover by balls of radius epsilon. Since A is strongly-fibred, for each $U \in \mathcal{U}$ there is $\rho \in \Sigma^\infty$ such that $A_\rho \subset U$. Hence, a point of $C_\infty(x_0, \sigma)$ lies in each $U \in \mathcal{U}$, by Theorem 1. It readily follows that $C_\infty(x_0, \sigma) \supset A$. But $C_\infty(x_0, \sigma) \subset A$; hence $C_\infty(x_0, \sigma) = A$. □

Note that Theorem 1 is stronger than Corollary 1.

Here we digress slightly from our main themes to reflect on the name "chaos game", since the process underlying the chaos game algorithm can be purely deterministic and does not need to be related in any way to ergodicity (e.g. Example 3). In dynamical systems theory the "furthest island" of stability is usually considered to be almost periodic behaviour, after stationary, periodic and quasi-periodic; beyond quasi-periodicity is the "ocean" of chaos. Following [30] we recall that an infinite sequence of symbols ς is **almost periodic** (or uniformly recursive) if, given any finite word τ that occurs in ς infinitely often we can associate a positive integer m such that any segment in ς of length m contains τ as a substring. Obviously

a disjunctive sequence cannot be almost periodic. Therefore the descriptive term "chaos game" retains its interpretation.

4. CATEGORIAL ANALYSIS

Subset $\Psi \subset M$ of a metric space M is called **porous** when

$$(4.1) \quad \exists_{0 < \lambda' < 1} \exists_{r_0 > 0} \forall_{\psi \in \Psi} \forall_{0 < r < r_0} \exists_{v \in M} N_{\lambda' r}\{v\} \subset N_r\{\psi\} \setminus \Psi.$$

A countable union of porous sets is said to be **σ -porous**. A subset of a σ -porous set is σ -porous.

Note that every σ -porous set is of the first Baire category and that this is a proper inclusion. Moreover every σ -porous subset of Euclidean space has null Lebesgue measure. In general metric spaces one can also relate the ideal of porous sets to the ideal of null sets under suitable assumptions. We quote such a result next and then show its natural application in Example 4.

Theorem 2 ([29] Propositions 3.5 & 3.3). *Let μ be the completion of a Borel regular probability measure on a separable metric space M which satisfies the doubling condition*

$$(4.2) \quad \exists_{r_0, c > 0} \forall_{\psi \in M} \forall_{0 < r < r_0} \mu(N_{2r}\{\psi\}) \leq c \cdot \mu(N_r\{\psi\}).$$

If $\Psi \subset M$ is σ -porous set, then it is null $\mu(\Psi) = 0$.

We remark that the regularity assumption is superfluous since probabilistic Borel measures on metric spaces are always regular ([9] Theorem 1.1) and completion adds only subsets of null sets. Fulfilling doubling condition everywhere implies that the measure is strictly positive (i.e., nonempty open sets have positive measure); thus the support of the measure is the whole space.

More on porosity can be found in [37, 29]. The book [24] uses porosity to study generics in optimization problems (cf. [10]). Results relating to porosity in fractal geometry and analysis can be found for example in [12, 25].

The following criterion will be useful.

Proposition 2. *If $\Psi \subset M$ satisfies*

$$(4.3) \quad \exists_{0 < \lambda < 1} \forall_{\psi \in \Psi} \forall_{n \geq 1} \exists_{v \in M} N_{\lambda \cdot 2^{-n}}\{v\} \subset N_{2^{-n}}\{\psi\} \setminus \Psi,$$

then Ψ is porous.

Proof. Choose $r_0 := 1$ and associate with $0 < r < r_0$ the number $n \geq 1$ in such a way that

$$2^{-n} < r \leq 2 \cdot 2^{-n}.$$

(Namely $n := \text{entier}[\log_2(r^{-1})] + 1$).

From (4.3) there exist appropriate $0 < \lambda < 1$ and $v \in M$. Scale $\lambda' := \frac{\lambda}{2}$ verifies (4.1):

$$N_{\lambda' r}\{v\} \subset N_{\lambda 2^{-n}}\{v\} \subset N_{2^{-n}}\{\psi\} \setminus \Psi \subset N_r\{\psi\} \setminus \Psi.$$

□

Now we recall that the **Cantor space** (Σ^∞, ϱ) is the set of infinite words over alphabet Σ equipped with the **Baire metric**

$$\varrho((\sigma_i)_{i=1}^\infty, (v_i)_{i=1}^\infty) := 2^{-\min\{i : \sigma_i \neq v_i\}}$$

for $(\sigma_i)_{i=1}^\infty, (v_i)_{i=1}^\infty \in \Sigma^\infty$ (conveniently $2^{-\min\emptyset} := 0$). Note that this space (Σ^∞, ϱ) may be referred to as **code space** in fractal geometry settings.

The topology of the Cantor space is just the Tikhonov product of the discrete alphabet Σ and so it is compact. But the Baire metric obeys ultrametric triangle inequality; this provides a tree structure in the space (compare also König's lemma on trees). The Cantor space appears among others in automata theory (e.g., [11] and references therein) and symbolic dynamics ([1], [2]).

For future reference we note that balls in the Baire metric are cylinders

$$(4.4) \quad \forall_{n \geq 1} \forall_{2^{-(n+1)} < r \leq 2^{-n}} \forall_{\psi = (\psi_i)_{i=1}^{\infty} \in \Sigma^{\infty}} N_r\{\psi\} = \{\psi_1\} \times \dots \times \{\psi_n\} \times \Sigma^{\infty}.$$

For $\tau = (\tau_1, \dots, \tau_m) \in \Sigma^m$ and $p \geq 1$ denote

$$\Psi(\tau, p) := \{(\sigma_i)_{i=1}^{\infty} \in \Sigma^{\infty} : \exists_{k \geq p} \forall_{l=1, \dots, m} \tau_l = \sigma_{(k-1)+l}\},$$

the set of words that do not contain the subword τ from the p -th position onwards.

Lemma 6. *The set $\Psi(\tau, p)$, as a subset of the code space $(\Sigma^{\infty}, \varrho)$, is a Borel set and porous.*

Proof. To simplify notation $\Psi := \Psi(\tau, p)$ and $\tilde{n} := n + p$ given $n \geq 1$.

Let $\psi = (\psi_i)_{i=1}^{\infty} \in \Psi$. We investigate $N_{2^{-n}}\{\psi\} \setminus \Psi$.

Define for $i \geq 1$

$$v_i := \begin{cases} \psi_i, & i < \tilde{n}, \\ \tau_{(i-\tilde{n}) \bmod m+1}, & i \geq \tilde{n}. \end{cases}$$

Of course $v := (v_i)_{i=1}^{\infty} \in \Sigma^{\infty} \setminus \Psi$. Moreover $v \in N_{2^{-n}}\{\psi\}$, because

$$\varrho(v, \psi) < 2^{-\tilde{n}} < 2^{-n}.$$

Consider $\varsigma = (\sigma_i)_{i=1}^{\infty} \in \Sigma^{\infty}$ close enough to v , namely

$$\varrho(\varsigma, v) < 2^{-(2m+p)} \cdot 2^{-n}.$$

Then $\sigma_i = v_i$ for $i \leq (2m+p) + n$. So

$$p < \tilde{n} + m < \tilde{n} + m + 1 < \dots < \tilde{n} + m + (m - 1) < 2m + p + n$$

and thus $\sigma_{\tilde{n}+m+l-1} = \tau_l$ for $l = 1, 2, \dots, m$, which in turn means that $\varsigma \notin \Psi$. Additionally

$$\varrho(\varsigma, \psi) \leq \varrho(\varsigma, v) + \varrho(v, \psi) < 2^{-1} \cdot 2^{-n} + 2^{-1} \cdot 2^{-n} = 2^{-n},$$

which means $\varsigma \in N_{2^{-n}}\{\psi\}$. Altogether

$$N_{\lambda \cdot 2^{-n}}\{v\} \subset N_{2^{-n}}\{\psi\} \setminus \Psi,$$

if we put $\lambda := 2^{-(2m+p)}$. Therefore Ψ is porous subject to condition (4.3).

The complement

$$\begin{aligned} \Sigma^{\infty} \setminus \Psi &= \bigcup_{k \geq 1} \Sigma^{p+(k-1)} \times \{\tau_1\} \times \dots \times \{\tau_m\} \times \Sigma^{\infty} = \\ &= \bigcup_{k \geq 1} \bigcup_{\pi \in \Sigma^{p+k-1}} N_{2^{-(p+k-1+m)}}\{\pi \cdot \tau\} \end{aligned}$$

is a countable union of open balls due to (4.4), hence Ψ is Borel. \square

Theorem 3. *Sequences which are not disjunctive form a Borel σ -porous set $D' \subset \Sigma^{\infty}$ w.r.t. the Baire metric.*

Proof. We have

$$\begin{aligned} D' &= \{(\sigma_i)_{i=1}^{\infty} \in \Sigma^{\infty} : (\sigma_i)_{i=1}^{\infty} \text{ does not obey condition (3.2)}\} = \\ &= \bigcup_{p \geq 1} \bigcup_{m \geq 1} \bigcup_{\tau \in \Sigma^m} \Psi(\tau, p). \end{aligned}$$

Since our union is countable, it is enough to remind that the sets $\Psi(\tau, p)$ are porous according to Lemma 6. \square

We are ready to prove the main theorem of this section.

Theorem 4. *The set of sequences $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$, which fail to generate a random orbit that yields the strongly-fibred attractor of the IFS F via (3.3) and (3.4) is σ -porous in $(\Sigma^{\infty}, \varrho)$.*

Proof. The set of faulty sequences is a subset of D' in Theorem 3. \square

5. PROBABILISTIC ANALYSIS

Let $Z_n : (S, \mathfrak{S}, \Pr) \rightarrow \Sigma$, $n = 1, 2, \dots$, be a sequence of random variables on a probability space (S, \mathfrak{S}, \Pr) , where \mathfrak{S} is a σ -algebra of events in S , and $\Pr : \mathfrak{S} \rightarrow [0, 1]$ probability measure. This stochastic process generates "truly" random sequences $(\sigma_1, \sigma_2, \dots) \in \Sigma^{\infty}$ i.e. $\sigma_n = Z_n(s)$ if the event $s \in S$ happens at the n -th stage.

We define the stochastic process $(Z_n)_{n \geq 1}$ to be **disjunctive** when

$$\forall_{m \geq 1} \forall_{\tau \in \Sigma^m} \Pr(Z_{(n-1)+l} = \tau_l, l = 1, \dots, m, \text{ for some } n) = 1;$$

that is, each finite word appears in the outcome with probability 1.

In fact all words almost surely appear infinitely often. But an even stronger assertion is true.

Proposition 3. *A disjunctive stochastic process $(Z_n)_{n \geq 1}$ with values in Σ generates a disjunctive sequence $(\sigma_n)_{n=1}^{\infty} \in \Sigma^{\infty}$ as its outcome with probability 1.*

Proof. Denote for $u \in \Sigma^m$

$$E(u) := \{(Z_{(n-1)+1}, \dots, Z_{(n-1)+m}) = u \text{ for some } n\}.$$

Define inductively $\gamma(p)$ to be the finite Champernowne word (Example 1) consisting of all finite words over Σ with length at most $p \geq 1$, and such that $\gamma(p+1)$ is just $\gamma(p)$ with attached at its end all finite words of length $p+1$. Thus the sequence of events $E(\gamma(p))$, $p = 1, 2, \dots$, is descending. Moreover by disjunctiveness of the process $\Pr(E(\gamma(p))) = 1$, so

$$\Pr \left(\bigcap_{p \geq 1} E(\gamma(p)) \right) = 1.$$

The event

$$\bigcap_{m \geq 1} \bigcap_{u \in \Sigma^m} E(u)$$

describes the appearance of a disjunctive sequence as an outcome. Its probability equals 1, because

$$\bigcap_{m \leq p} \bigcap_{u \in \Sigma^m} E(u) \supset E(\gamma(p)).$$

\square

Example 2. (*Bernoulli scheme; [1]*). Suppose $(Z_n)_{n \geq 1}$ is the sequence of independent random variables is distributed according to

$$(5.1) \quad \exists_{\alpha > 0} \forall_{n \geq 1} \forall_{\sigma \in \Sigma} \Pr(Z_n = \sigma) \geq \alpha.$$

An example is the classical Bernoulli scheme with outcomes in Σ . Then $(Z_n)_{n \geq 1}$ is disjunctive process. This follows from the Borel-Cantelli lemma (e.g. the classic Example on p.37 after Theorem 2.2.3 in [21]).

For Bernoulli scheme one could alternatively apply Theorem 2.3 (item 6) from [11] which says that the set of nondisjunctive sequences is null with respect to the Bernoulli product measure. This follows as corollary from combination of Theorems 3 and 2. See Example 4 below for a more general case.

Although ergodic stochastic processes are useful in engineering applications (e.g. [15, 16, 34]) they might be too weak for reliable simulations in probabilistic algorithms like the chaos game. (In particular, pseudorandom number generators that pass a battery of statistical tests may fail to generate an attractor).

Example 3. (*Ergodicity is not enough; [23] Example 1.8.1*). Let $(Z_n)_{n \geq 1}$ be the homogeneous Markov chain with states in $\Sigma := \{1, 2\}$ such that

$$\begin{aligned} \forall_{\sigma \in \Sigma} \Pr(Z_1 = \sigma) &= \frac{1}{2}, \\ \forall_{n \geq 2} \Pr(Z_n = 1 | Z_{n-1} = 2) &= 1, \\ \forall_{n \geq 2} \forall_{\sigma \in \Sigma} \Pr(Z_n = \sigma | Z_{n-1} = 1) &= \frac{1}{2}. \end{aligned}$$

(Note that we put also condition on initial distribution of the chain). It is ergodic (even strongly mixing as the square of its transition matrix has positive entries; e.g.[34] Prop.I.2.10). Moreover our chain occupies all states almost surely:

$$\forall_{\sigma \in \Sigma} \Pr(Z_n = \sigma \text{ for infinitely many } n) = 1.$$

Nevertheless the word "22" is forbidden:

$$\Pr(Z_n = 2, Z_{n+1} = 2 \text{ for some } n) = 0,$$

i.e. the process lacks disjunctiveness (comp. with discussion in [34] chap.I.4).

In relation with Example 4 it is not hard to see that a homogeneous finite Markov chain (with strictly positive initial distribution) is disjunctive if and only if its transition matrix has positive entries.

Example 4. (*Chain with complete connections; [6]*). Let $(Z_n)_{n \geq 1}$ be a sequence of random variables with conditional marginal distributions

$$(5.2) \quad \exists_{\alpha > 0} \forall_{n \geq 1} \forall_{\sigma_1, \dots, \sigma_n \in \Sigma} \Pr(Z_n = \sigma_n | Z_{n-1} = \sigma_{n-1}, \dots, Z_1 = \sigma_1) \geq \alpha,$$

and initial distribution

$$\Pr(Z_1 = \sigma_1) \geq \alpha.$$

Sometimes it is called **chain with complete connections** and significantly generalizes usual Markov chain ([19]). We shall indirectly prove that such minorized chains are disjunctive processes.

Define on (Σ^∞, ϱ) probabilistic measure μ to be the completion of the joint distribution of the process $(Z_n)_{n \geq 1}$:

$$\mu(\Xi) := \Pr((Z_1, Z_2, \dots) \in \Xi)$$

for Borel subsets $\Xi \subset \Sigma^\infty$ (comp. [34] Theorem I.1.2 or [15] Section 2.7 and Chapter 3). By description of balls given in (4.4)

$$\mu(N_r\{\varsigma\}) = \Pr(Z_1 = \sigma_1, \dots, Z_n = \sigma_n)$$

for radii $r \in (2^{-(n+1)}, 2^{-n}]$, $n \geq 1$, and centers at $\varsigma = (\sigma_1, \dots, \sigma_n, \dots) \in \Sigma^\infty$.

The measure μ obeys doubling condition. Indeed assume now $2r \in (2^{-(n+1)}, 2^{-n}]$ and calculate

$$\begin{aligned} \mu(N_{2r}\{\varsigma\}) &= \Pr(Z_1 = \sigma_1, \dots, Z_n = \sigma_n) = \\ &= \frac{\Pr(Z_1 = \sigma_1, \dots, Z_{n+1} = \sigma_{n+1})}{\Pr(Z_{n+1} = \sigma_{n+1} \mid Z_n = \sigma_n, \dots, Z_1 = \sigma_1)} \leq \frac{1}{\alpha} \cdot \mu(N_r\{\varsigma\}), \end{aligned}$$

where the inequality comes from (5.2).

Therefore we can apply Theorem 2 and Theorem 3 to find out that nondisjunctive sequences form Borel μ -null set, so the chain generates disjunctive sequence almost surely.

We finalize this section by giving its main result, which follows directly from Theorem 1 via Proposition 3.

Theorem 5. Let A be a strongly-fibred attractor of the IFS $F = (X, f_\sigma : \sigma \in \Sigma)$. If the stochastic process $Z_n : (S, \mathfrak{S}, \Pr) \rightarrow \Sigma$, $n = 1, 2, \dots$, generating $(\sigma_n)_{n=1}^\infty \in \Sigma^\infty$ is disjunctive, then (3.3) and (3.4) in the statement of Corollary 1 hold with probability 1.

6. THE RAPUNZEL THEOREM

Let $F = (X, f_\sigma : \sigma \in \Sigma)$ be an IFS of homeomorphisms acting on a compact metric space X . Let

$$F^* := (X, f_\sigma^{-1} : \sigma \in \Sigma)$$

be the corresponding **dual** IFS. Let A be an attractor of F , and let $\mathcal{B}(A)$ denote the basin of A . Then the set $A^* := X \setminus \mathcal{B}(A)$ is called the **dual repeller** and (A, A^*) is called an **attractor/repeller** pair. We suppose here that A^* is an attractor of F^* . It is readily proved that the basin $\mathcal{B}(A^*)$ of A^* (with respect to F^*) is $\mathcal{B}(A^*) = X \setminus A$. Note that $\mathcal{B}(A) = X \setminus A^*$. We furthermore suppose that A is **point-fibred** with respect to F and A^* is point-fibred with respect to F^* , see [28, Chapter 4]. This means that there exist continuous maps

$$\pi_F : \Sigma^\infty \rightarrow A \text{ and } \pi_{F^*} : \Sigma^\infty \rightarrow A^*$$

that are well-defined for $\varsigma = \sigma_1 \dots \sigma_k \dots \in \Sigma^\infty$ by

$$\pi_F(\varsigma) = \lim_{k \rightarrow \infty} f_{\sigma_1} \circ \dots \circ f_{\sigma_k}(x), \quad x \in B,$$

$$\pi_{F^*}(\varsigma) = \lim_{k \rightarrow \infty} f_{\sigma_1}^{-1} \circ \dots \circ f_{\sigma_k}^{-1}(y), \quad y \in B^*,$$

where the limits are independent of x and y . Moreover we have for all $\varsigma \in \Sigma^\infty$

$$(6.1) \quad \pi_F(S(\varsigma)) = f_{\sigma_1}^{-1}(\pi_F(\varsigma)) \text{ and } \pi_{F^*}(S(\varsigma)) = f_{\sigma_1}(\pi_{F^*}(\varsigma))$$

where $S : \Sigma^\infty \rightarrow \Sigma^\infty$ is the shift map, namely the continuous mapping defined by

$$S(\varsigma) = \sigma_2 \sigma_3 \dots \text{ for all } \varsigma = \sigma_1 \sigma_2 \sigma_3 \dots \in \Sigma^\infty.$$

In general, an IFS of homeomorphisms can have many attractor/repeller pairs. Here we are considering only the situation where F has exactly one attractor.

Our terminology and ideas derive from [26] and [27]. However, there is a crucial difference in nomenclature, because what McGehee calls an "attractor" we would call a "Conley attractor".

Theorem 6. *Let F be an IFS of homeomorphisms with a unique point-fibred attractor A and point-fibred dual repeller A^* . Let ς be a disjunctive sequence. Then there is a set of points $X' \subset X$ such that (i) $X \setminus X'$ is σ -porous; (ii) the chaos game orbit generated by F, x, ς yields A for all $x \in X'$; (iii) the dual chaos game orbit generated by F^*, x, ς yields A^* for all $x \in X'$.*

Proof. Let $x \in X$. If $x = \pi_{F^*}(\varsigma)$ then, given any open neighbourhood $O(x)$ of x there is an open set $O(x') \subset O(x) \setminus \{x\}$ and, obviously, every point y in $O(x')$ either belongs to the basin $\mathcal{B}(A)$ of A , in which case its orbit yields A , or $y \in A^*$ and has a compact set of addresses $\pi_{F^*}^{-1}(y)$ that does not include ς . Let K be the highest index of agreement between ς and any member of $\pi_{F^*}^{-1}(y)$. Then, using equation (6.1), we must have

$$f_{\sigma_{K+1}}^{-1} \circ f_{\sigma_K}^{-1} \circ f_{\sigma_{K-1}}^{-1} \circ \dots \circ f_{\sigma_1}^{-1}(y) \in \mathcal{B}(A),$$

(for otherwise there would have been one higher level of agreement) which tells us (using disjunctiveness) that the chaos game orbit generated by F, y, ς yields A . It follows that the set of points x , denoted X'' , for which the chaos game generated by F, x, ς yields A has a σ -porous complement $X \setminus X''$. Similarly for x in the set of points Y , for which the chaos game generated by F^*, x, ς yields A^* has also a σ -porous complement. The proof is completed by choosing $X' = X'' \cap Y$. \square

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